Symmetric Teleparallel Horndeski

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Outline

- Symmetric teleparallel gravity
- Symmetric teleparallel Horndeski: scheme
- Systematic construction
- Cosmology
- Conclusion

Geometric preliminaries

J.D. McCrea 92.

- Fundamental object: non-metricity tensor $Q_{\lambda\mu\nu} \equiv \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho}$

- Irreducible pieces: $Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda + \not{Q}_{\lambda\mu\nu}$

Weyl part

$$W_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu}$$

„traceless" part

$$\not{Q}_{\lambda\mu\nu} = g_{\lambda(\mu}\Lambda_{\nu)} - \frac{1}{4} g_{\mu\nu}\Lambda_\lambda + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu}$$

where

$$\Lambda_\mu = \frac{4}{9}\left(Q^\nu{}_{\mu\nu} - W_\mu\right),$$

$$\Omega_\lambda{}^{\mu\nu} = -\left[\varepsilon^{\mu\nu\rho\sigma} Q_{\rho\sigma\lambda} + \varepsilon^{\mu\nu\rho}{}_{\lambda}\left(\frac{3}{4}\Lambda_\rho - W_\rho\right)\right]$$

$$q_{\lambda\mu\nu} = Q_{(\lambda\mu\nu)} - g_{(\mu\nu}W_{\lambda)} - \frac{3}{4} g_{(\mu\nu}\Lambda_{\lambda)},$$

M. Adak et al. '06, J. B. Jiménez et al. '17

Symmetric teleparallel (ST) gravity

- No torsion: $T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0$ ➡ $\Gamma^\alpha{}_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\lambda \partial_\mu \Lambda^\lambda{}_\nu$
- No curvature: $R^\alpha{}_{\rho\mu\nu} \equiv \partial_\mu \Gamma^\alpha{}_{\nu\rho} - \partial_\nu \Gamma^\alpha{}_{\mu\rho} + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\mu\rho} = 0$ ➡ $\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\lambda}\partial_\mu\partial_\nu \xi^\lambda$
- Can relate $R^\sigma{}_{\rho\mu\nu} = \mathring{R}^\sigma{}_{\rho\mu\nu} - \mathring{\nabla}_\nu L^\sigma{}_{\mu\rho} + \mathring{\nabla}_\mu L^\sigma{}_{\nu\rho} - L^\sigma{}_{\nu\lambda} L^\lambda{}_{\mu\rho} + L^\sigma{}_{\mu\lambda} L^\lambda{}_{\nu\rho} = 0$
- Coincident gauge $\xi^\mu = x^\mu \quad \Rightarrow \quad \Gamma^\alpha{}_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_\mu = \partial_\mu$
- Einstein-Hilbert action becomes "Einstein action"; diff-invariance lost.

$$S_{\mathrm{STEGR}}|_{\Gamma=0} = \frac{M_{\mathrm{pl}}^2}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\left(\mathring{\Gamma}^\alpha{}_{\beta\mu}\mathring{\Gamma}^\beta{}_{\nu\alpha} - \mathring{\Gamma}^\alpha{}_{\beta\alpha}\mathring{\Gamma}^\beta{}_{\mu\nu}\right)$$

2 formulations of GR

J. B. Jiménez et al. ‘19

- Riemannian GR: Curvature $R^{\alpha}{}_{\rho\mu\nu} = \mathring{R}^{\alpha}{}_{\rho\mu\nu} \neq 0$ $\quad T^{\lambda}{}_{\mu\nu} = 0$ $\quad Q_{\lambda\mu\nu} = 0$

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \mathring{R}$$

➡ 10 - 2x4 = 2 propagating d.o.f

- STEGR: Non-metricity

$$S_{\rm Newer\ GR} = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, \mathbb{Q}$$

Equivalently could be expressed in terms of

$$Q_1 = W^{\mu} W_{\mu}$$
$$Q_2 = \Lambda_{\mu} \Lambda^{\mu}$$
$$Q_3 = W_{\mu} \Lambda^{\mu}$$
$$Q_4 = *\Omega_{\alpha\mu\nu} *\Omega^{\alpha\mu\nu}$$
$$Q_5 = q_{\lambda\mu\nu} q^{\lambda\mu\nu}$$

⬅

$$\mathbb{Q} = c_1 Q_{\alpha}{}^{\mu\nu} Q^{\alpha}{}_{\mu\nu} + c_2 Q_{\alpha}{}^{\mu\nu} Q_{\mu}{}^{\alpha}{}_{\nu} + c_3 Q_{\mu} Q^{\mu} + c_4 \tilde{Q}_{\mu} \tilde{Q}^{\mu} + c_5 \tilde{Q}_{\mu} Q^{\mu}$$

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{2}, \quad c_3 = -\frac{1}{4}, \quad c_4 = 0, \quad c_5 = \frac{1}{2}$$

➡ Non-metricity scalar

$$\mathring{R} = -Q + \mathring{\nabla}_{\mu}(\tilde{Q}^{\mu} - Q^{\mu}) := -Q + B_Q$$

➡ Equivalent up to boundary terms

ST Horndeski - conditions

- Euler-Lagrange (EL) equations for all the fields $(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \phi)$ at most second order

  ➡ No Ostrogradsky ghosts

- Lagrangian is parity preserving ➡ useful to define $*\Omega^{\alpha\mu\nu} = \epsilon^{\rho\sigma\mu\nu}\Omega^{\alpha}{}_{\rho\sigma}$

- Invariants at most quadric in the non-metricity $Q_{\lambda\mu\nu}$

- Max. # of d.o.f. = 11 (easy to see in coincident gauge)

Covariantization & variations

- Covariantization prescription: $\eta_{\mu\nu} \to g_{\mu\nu}$

  $\partial_\mu \to \mathring{\nabla}_\mu$

- In ST-geometry: in principle other prescription possible: $\eta_{\mu\nu} \to g_{\mu\nu}$

  $\partial_\mu \to \nabla_\mu$

➡ Matter couples non-minimally

- Variation w.r.t. the metric, the scalar field and the flat connection

- 2 ways of ST extensions:
  a) Dependence of G_i functions on $Q_{\lambda\mu\nu}$ (1^{st} order derivatives)

  b) Derivative terms $\mathring{\nabla}_\alpha Q_{\lambda\mu\nu}$ in Lagrangian (2^{nd} order derivatives)

ST Horndeski - scheme

- General schematic form:

(Dimensionfull) coefficients

Up to 2nd derivatives in ϕ

Up to 1st derivatives in $\mathcal{Q}$

$$L_i = \sum_{\substack{N_\phi, N_Q \geq 0 \\ N_\phi \geq n \geq 0}} C_{N_\phi, N_Q, n} \left[\sum_{\substack{m,r,l \geq 0 \\ m+2r+l=i-2}} A^{(n,N_\phi,N_Q)}_{m,r,l} \, \phi^{N_\phi - n - m} (\partial\phi)^n \, (\mathring{\nabla}\mathring{\nabla}\phi)^m \, \mathring{R}^r \, \mathcal{Q}^{N_Q - l} \, (\mathring{\nabla}\mathcal{Q})^l \right]$$

Curvature associated to LC-conn.

Non-metricity building blocks: $\mathcal{Q} = (W^\mu, \Lambda^\mu, q^{\mu\nu\rho}, *\Omega^{\alpha\mu\nu})$

- 2nd order EL eqs. not guaranteed yet

- Full Lagrangian: $L = \sum_{i=2} c_i L_i$

ST Horndeski – special cases

- Q only: $$L = \sum_{N_Q \geq 0} C_{N_Q}\, Q^{N_Q} \equiv f(\{Q\})$$ Invariants quadratic in Q ➡ $$L = f(Q, Q_1, Q_2, Q_3, Q_4)$$

  $N_\varphi = 0$

  $i = 2$

- Riemannian Horndeski: $$\mathring{L}_i = \sum_{N_\phi \geq n \geq 0} C_{N_\phi, n} \left[\sum_{\substack{m,r\geq 0 \\ m+2r=i-2}} A^{(n,N_\phi)}_{m,r}\, \phi^{N_\phi - n - m} (\partial\phi)^n\, (\mathring{\nabla}\mathring{\nabla}\phi)^m\, \mathring{\mathcal{R}}^r \right]$$

  $N_Q = 0$

  $2 \leq i \leq 5$

L_2– systematic construction (no HD)

- General form: $$L_2 = \sum_{N_\phi, N_Q \geq 0} \sum_{n=0}^{N_\phi} c_{N_\phi, N_Q, n}\, \phi^{N_\phi - n} (\partial\phi)^n \mathcal{Q}^{N_Q} \equiv L_2(\phi, \partial\phi, \{\mathcal{Q}\})$$ $$\mathcal{Q} = (W^\mu, \Lambda^\mu, q^{\mu\nu\rho}, *\Omega^{\alpha\mu\nu})$$

n=1
$$\begin{aligned} I_1 &= W^\mu \phi_{;\mu}\,, \\ I_2 &= \Lambda^\mu \phi_{;\mu}\,, \end{aligned}$$

n=2
$$\begin{aligned} J_1 &= q_{\lambda\mu\nu} W^\lambda \phi^{;\mu}\phi^{;\nu}\,, \\ J_2 &= q_{\lambda\mu\nu} \Lambda^\lambda \phi^{;\mu}\phi^{;\nu}\,, \\ J_3 &= *\Omega_\mu{}^{\nu\sigma} W_\sigma \phi^{;\mu}\phi_{;\nu}\,, \\ J_4 &= *\Omega_\mu{}^{\nu\sigma} \Lambda_\sigma \phi^{;\mu}\phi_{;\nu}\,, \\ J_5 &= q_{\lambda\alpha\mu} q^{\lambda\alpha}{}_\nu \phi^{;\mu}\phi^{;\nu}\,, \\ J_6 &= q_{\lambda\alpha\mu} *\Omega^{\lambda\alpha}{}_\nu \phi^{;\mu}\phi^{;\nu} \\ J_7 &= *\Omega_{\lambda\alpha\mu} *\Omega^{\lambda\alpha}{}_\nu \phi^{;\mu}\phi^{;\nu}\,, \\ J_8 &= *\Omega_{\lambda\alpha\mu} *\Omega_\nu{}^{\lambda\alpha} \phi^{;\mu}\phi^{;\nu}\,. \end{aligned}$$

n=3
$$J_9 = q_{\lambda\mu\nu} \phi^{;\lambda}\phi^{;\mu}\phi^{;\nu}\,,$$

n=4
$$\begin{aligned} J_{10} &= q_{\mu\nu\sigma} q_{\lambda\alpha}{}^\sigma \phi^{;\mu}\phi^{;\nu}\phi^{;\lambda}\phi^{;\alpha}\,, \\ J_{11} &= q_{\mu\nu\sigma} *\Omega_{\lambda\alpha}{}^\sigma \phi^{;\mu}\phi^{;\nu}\phi^{;\lambda}\phi^{;\alpha}\,, \\ J_{12} &= *\Omega_{\mu\nu\sigma} *\Omega_{\lambda\alpha}{}^\sigma \phi^{;\mu}\phi^{;\nu}\phi^{;\lambda}\phi^{;\alpha}\,. \end{aligned}$$

$$L_2 = \mathring{L}_2 + L_{2-\mathrm{STele}} = G_2(\phi, X) + G_{\mathrm{ST}}(Q, Q_1, Q_2, Q_3, Q_4, \phi, X, I_1, I_2, J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9, J_{10}, J_{11}, J_{12})$$

L_3 – systematic construction (HD)

General form:

$$L_3 = \sum_{\substack{N_\phi, N_Q \geq 0 \\ N_\phi \geq n \geq 0}} C_{N_\phi, N_Q, n} \left[\sum_{\substack{m,l \geq 0 \\ m+l=1}} A_{m,l}^{(n,N_\phi,N_Q)} \, \phi^{N_\phi - n - m} (\partial\phi)^n \, (\mathring{\nabla}\mathring{\nabla}\phi)^m \, Q^{N_Q - l} \, (\mathring{\nabla} Q)^l \right]$$

$N_Q = 0$

$$\mathring{L}_3 = L_3(N_Q = 0) = \sum_{\substack{N_\phi \geq 0 \\ N_\phi \geq n \geq 0}} C_{N_\phi, N_Q=0, n} \left[A_{1,0}^{(n,N_\phi,N_Q=0)} \, \phi^{N_\phi - n - 1} (\partial\phi)^n \, (\mathring{\nabla}\mathring{\nabla}\phi) \right]$$

$$= -G_3(\phi, X) \, \mathring{\Box}\phi \, ,$$

➡ Riemannian KGB

$N_Q = 1$

$$L_3(N_Q = 1) = \sum_{\substack{N_\phi \geq 0 \\ N_\phi \geq n \geq 0}} C_{N_\phi, N_Q=1, n} \left[A_{1,0}^{(n,N_\phi,N_Q=1)} \, \phi^{N_\phi - n - 1} (\partial\phi)^n \, (\mathring{\nabla}\mathring{\nabla}\phi) \, Q + A_{0,1}^{(n,N_\phi,N_Q=1)} \phi^{N_\phi - n} (\partial\phi)^n \, (\mathring{\nabla} Q) \right]$$

$$= \sum_a \left[\tilde{G}_3^{(a)}(\phi, X) \tilde{\mathcal{O}}_a + F_3^{(a)}(\phi, X) \hat{\mathcal{O}}_a \right] ,$$

➡ 2 types of contractions;
not independent.

L_3 – systematic construction (HD)

- All possible contractions linear in Q and $\mathring{\nabla}\mathring{\nabla}\phi$ with increasing factors of $\mathring{\nabla}\phi$:

$$
\begin{aligned}
\tilde{\mathcal{O}}_1 &= \phi^{;\mu} Q_{\mu\nu}{}^{\nu}\, \mathring{\Box}\phi &&= 4\mathcal{O}_{W1}\,,\\
\tilde{\mathcal{O}}_2 &= \phi^{;\mu} Q_{\nu\mu}{}^{\nu}\, \mathring{\Box}\phi &&= \frac{9}{4}\mathcal{O}_{\Lambda 1} + \mathcal{O}_{W1}\,,\\
\tilde{\mathcal{O}}_3 &= \phi_{;\alpha} Q_{\beta\mu}{}^{\mu}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi &&= 4\mathcal{O}_{W2}\,,\\
\tilde{\mathcal{O}}_4 &= \phi_{;\alpha} Q_{\mu\beta}{}^{\mu}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi &&= \frac{9}{4}\mathcal{O}_{\Lambda 2} + \mathcal{O}_{W2}\,,\\
\tilde{\mathcal{O}}_5 &= \phi^{;\mu} Q_{\alpha\mu\beta}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi &&= \mathcal{O}_{W2} + \frac{1}{4}\mathcal{O}_{\Lambda 2} + \frac{1}{2}\mathcal{O}_{\Lambda 1} + \frac{1}{6}\mathcal{O}_{\Omega 1} + \mathcal{O}_{q1}\,,\\
\tilde{\mathcal{O}}_6 &= \phi^{;\mu} Q_{\mu\alpha\beta}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi &&= \mathcal{O}_{W1} + \mathcal{O}_{\Lambda 2} - \frac{1}{4}\mathcal{O}_{\Lambda 1} - \frac{1}{3}\mathcal{O}_{\Omega 1} + \mathcal{O}_{q1}\,,\\
\tilde{\mathcal{O}}_7 &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha} Q_{\mu\nu\alpha}\, \mathring{\Box}\phi &&= -2X\,\mathcal{O}_{W1} - \frac{3}{2}X\,\mathcal{O}_{\Lambda 1} + \mathcal{O}_{q2}\,,\\
\tilde{\mathcal{O}}_8 &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha} Q_{\nu\beta}{}^{\beta}\, \mathring{\nabla}_{\mu}\mathring{\nabla}_{\alpha}\phi &&= 4\mathcal{O}_{W3}\,,\\
\tilde{\mathcal{O}}_9 &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha} Q_{\beta\nu}{}^{\beta}\, \mathring{\nabla}_{\mu}\mathring{\nabla}_{\alpha}\phi &&= \frac{9}{4}\mathcal{O}_{\Lambda 3} + \mathcal{O}_{W3}\,,\\
\tilde{\mathcal{O}}_{10} &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha} Q_{\mu\nu}{}^{\beta}\, \mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi &&= \mathcal{O}_{W3} - X\,\mathcal{O}_{\Lambda 2} + \frac{1}{4}\mathcal{O}_{\Lambda 3} + \frac{1}{6}\mathcal{O}_{\Omega 2} + \mathcal{O}_{q3}\,,\\
\tilde{\mathcal{O}}_{11} &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha} Q^{\beta}{}_{\mu\nu}\, \mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi &&= -2X\,\mathcal{O}_{W2} + \mathcal{O}_{\Lambda 3} + \frac{1}{2}X\,\mathcal{O}_{\Lambda 2} - \frac{1}{3}\mathcal{O}_{\Omega 2} + \mathcal{O}_{q3}\,,\\
\tilde{\mathcal{O}}_{12} &= \phi^{;\mu}\phi^{;\nu}\phi^{;\alpha}\phi^{;\rho}\phi^{;\sigma} Q_{\mu\nu\alpha}\, \mathring{\nabla}_{\rho}\mathring{\nabla}_{\sigma}\phi &&= -2X\,\mathcal{O}_{W3} - \frac{3}{2}X\,\mathcal{O}_{\Lambda 3} + \mathcal{O}_{q4}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{O}_{W1} &= W_{\mu}\phi^{;\mu}\, \mathring{\Box}\phi\,,\\
\mathcal{O}_{W2} &= W_{\alpha}\phi_{;\beta}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\,,\\
\mathcal{O}_{W3} &= W_{\mu}\phi^{;\mu}\phi^{;\alpha}\phi^{;\beta}\, \mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi\,,\\
\mathcal{O}_{\Lambda 1} &= \Lambda_{\mu}\phi^{;\mu}\, \mathring{\Box}\phi\,,\\
\mathcal{O}_{\Lambda 2} &= \Lambda_{\alpha}\phi_{;\beta}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\,,\\
\mathcal{O}_{\Lambda 3} &= \Lambda_{\mu}\phi^{;\mu}\phi^{;\alpha}\phi^{;\beta}\, \mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi\,,\\
\mathcal{O}_{\Omega 1} &= *\Omega_{\alpha\beta\mu}\phi^{;\mu}\, \mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\,,\\
\mathcal{O}_{\Omega 2} &= *\Omega_{\alpha\beta\mu}\phi^{;\alpha}\phi^{;\beta}\phi_{;\nu}\, \mathring{\nabla}^{\mu}\mathring{\nabla}^{\nu}\phi\,,\\
\mathcal{O}_{q1} &= q_{\alpha\beta\mu}\phi^{;\mu}\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\,,\\
\mathcal{O}_{q2} &= q_{\alpha\beta\mu}\phi^{;\mu}\phi^{;\alpha}\phi^{;\beta}\, \mathring{\Box}\phi\,,\\
\mathcal{O}_{q3} &= q_{\alpha\beta\mu}\phi^{;\alpha}\phi^{;\beta}\phi_{;\nu}\, \mathring{\nabla}^{\mu}\mathring{\nabla}^{\nu}\phi\,,\\
\mathcal{O}_{q4} &= q_{\alpha\beta\mu}\phi^{;\mu}\phi^{;\alpha}\phi^{;\beta}\phi^{;\rho}\phi^{;\sigma}\, \mathring{\nabla}_{\rho}\mathring{\nabla}_{\sigma}\phi\,.
\end{aligned}
$$

L_3– systematic construction (HD)

- 5 Combinations that lead to 2nd order EL-eqs:

$$
\begin{aligned}
L_3^{(1)} &= \tilde{G}_3^{(1)}(\phi,X)(\tilde{\mathcal{O}}_3-\tilde{\mathcal{O}}_1) = 2\tilde{G}_3^{(1)}(\phi,X)\phi_{;\alpha}L^{\mu}{}_{\beta\mu}\left[\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi - g^{\alpha\beta}\mathring{\Box}\phi\right] ,\\
L_3^{(2)} &= \tilde{G}_3^{(2)}(\phi,X)(\tilde{\mathcal{O}}_4-\tilde{\mathcal{O}}_6) = \tilde{G}_3^{(2)}(\phi,X)\left(g_{\alpha\beta,\mu} - \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\beta} - \Gamma^{\lambda}{}_{\mu\beta}g_{\alpha\lambda}\right)\left[\phi_{;\rho}g^{\mu\alpha}\mathring{\nabla}^{\rho}\mathring{\nabla}^{\beta}\phi - \phi^{;\mu}\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\right] ,\\
L_3^{(3)} &= \tilde{G}_3^{(3)}(\phi,X)(\tilde{\mathcal{O}}_5-\tilde{\mathcal{O}}_2) = \tilde{G}_3^{(3)}(\phi,X)\phi^{;\mu}\left(g_{\mu\beta,\alpha} - \Gamma^{\lambda}{}_{\alpha\mu}g_{\lambda\beta} - \Gamma^{\lambda}{}_{\alpha\beta}g_{\mu\lambda}\right)\left[\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi - g^{\alpha\beta}\mathring{\Box}\phi\right] ,\\
L_3^{(4)} &= \tilde{G}_3^{(4)}(\phi,X)(\tilde{\mathcal{O}}_{10}-\tilde{\mathcal{O}}_9) = \tilde{G}_3^{(4)}(\phi,X)\phi^{;\alpha}\phi^{;\nu}\left(g_{\nu\rho,\mu} - \Gamma^{\lambda}{}_{\mu\nu}g_{\lambda\rho} - \Gamma^{\lambda}{}_{\mu\rho}g_{\nu\lambda}\right)\left[\phi^{;\mu}g^{\beta\rho} - \phi^{;\beta}g^{\rho\mu}\right]\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi ,\\
L_3^{(5)} &= \tilde{G}_3^{(5)}(\phi,X)(\tilde{\mathcal{O}}_{11}-\tilde{\mathcal{O}}_7) = \tilde{G}_3^{(5)}(\phi,X)\phi^{;\rho}\phi^{;\nu}\left(g_{\nu\rho,\mu} - \Gamma^{\lambda}{}_{\mu\nu}g_{\lambda\rho} - \Gamma^{\lambda}{}_{\mu\rho}g_{\nu\lambda}\right)\left[\phi^{;\alpha}g^{\mu\beta}\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi - \phi^{;\mu}\mathring{\Box}\phi\right] ,
\end{aligned}
$$

⬇

Pairwise compensation

$\tilde{\mathcal{O}}_8$ and $\tilde{\mathcal{O}}_{12}$ do not appear

Can rewrite in basis of irreducible pieces ➡

$$
\begin{aligned}
L_3^{(1)} &= (\mathcal{O}_{W2}-\mathcal{O}_{W1})G_3^{(1)}(\phi,X) := \mathcal{O}_1 G_3^{(1)}(\phi,X) ,\\
L_3^{(2)} &= (5\mathcal{O}_{\Lambda1}+\mathcal{O}_{\Lambda2}-4\mathcal{O}_{q1})\,G_3^{(2)}(\phi,X) := \mathcal{O}_2 G_3^{(2)}(\phi,X) ,\\
L_3^{(3)} &= \Big(3(\mathcal{O}_{\Lambda2}-\mathcal{O}_{\Lambda1})+\mathcal{O}_{\Omega1}\Big)G_3^{(3)}(\phi,X) := \mathcal{O}_3 G_3^{(3)}(\phi,X) ,\\
L_3^{(4)} &= \Big(3X(\mathcal{O}_{\Lambda2}-\mathcal{O}_{\Lambda1})+2(3\mathcal{O}_{\Lambda3}+\mathcal{O}_{q2})-6\mathcal{O}_{q3}\Big)G_3^{(4)}(\phi,X) := \mathcal{O}_4 G_3^{(4)}(\phi,X) ,\\
L_3^{(5)} &= \Big(3X(\mathcal{O}_{\Lambda2}+\mathcal{O}_{\Lambda1})+2(3\mathcal{O}_{\Lambda3}-\mathcal{O}_{q2})-\mathcal{O}_{\Omega2}\Big)G_3^{(5)}(\phi,X) := \mathcal{O}_5 G_3^{(5)}(\phi,X) .
\end{aligned}
$$

L_3– systematic construction (HD)

- Total Lagrangian:

$$
\begin{aligned}
L_{\text{STKGB}} &= \mathring{R} + \mathring{L}_2 + \mathring{L}_3 + \sum_{a=1}^{5} L_3^{(a)} \\
&= \mathring{R} + G_2(\phi, X) - G_3(\phi, X)\mathring{\Box}\phi + \mathcal{O}_1 G_3^{(1)}(\phi, X) + \mathcal{O}_2 G_3^{(2)}(\phi, X) + \mathcal{O}_3 G_3^{(3)}(\phi, X) \\
&\quad + \mathcal{O}_4 G_3^{(4)}(\phi, X) + \mathcal{O}_5 G_3^{(5)}(\phi, X) \,,
\end{aligned}
$$

- Recipe to go to higher order in Q, e.g. $N_Q = 2$ ➡ tedious

- Also could go to higher i, e.g. $i = 4$ expect terms like $\mathcal{Q}(\mathring{\nabla}\mathring{\nabla}\phi)^2$, $(\mathring{\nabla}\mathring{\nabla}\phi)\mathring{\nabla}\mathcal{Q}$

- Potentially more general dependency on non-metricity-invariants in the generic functions.
  ➡ N_Q arbitrary

Cosmology: flat FLRW

- Consider: $L_{\mathrm{ST-Horn}} = \sum_{i=4}^{5} \mathring{L}_i + G_{\mathrm{ST}}(Q_i, \phi, X, I_i, J_i) + L_{\mathrm{STKGB}}$
- Ensure metric and connection respect the same symmetries: Killing vector fields Z_ζ where $\zeta = \{1, .., m\}$

Solve $(\mathcal{L}_{Z_\zeta} g)_{\mu\nu} = 0$ $(\mathcal{L}_{Z_\zeta} \Gamma)^\lambda{}_{\mu\nu} = 0$

⬇

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 (dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2)$$

⬇

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \qquad n_\mu = (-N, 0, 0, 0)$$

⬇

$Q_{\rho\mu\nu} = 2F_1 n_\rho n_\mu n_\nu + 2F_2 n_\rho h_{\mu\nu} + 2F_3 h_{\rho(\mu} n_{\nu)}$ $F_i = F_i(t)$ ➡ 3 branches and $\Omega_{\lambda\mu\nu} = 0$

Conclusion

- Formulated scheme for potential extensions of Horndeski interactions in ST gravity
- Constructed the most general L_2
- Constructed an extension belonging to L_3 class up to $N_Q = 1$
- Future directions could include:
  - Check $c_T = 1$ constraints
  - Cosmological perturbations; strong coupling in f(Q) around one branch in cosmology
  - Scalarized BHs
  - Radiative stability

Thank you!