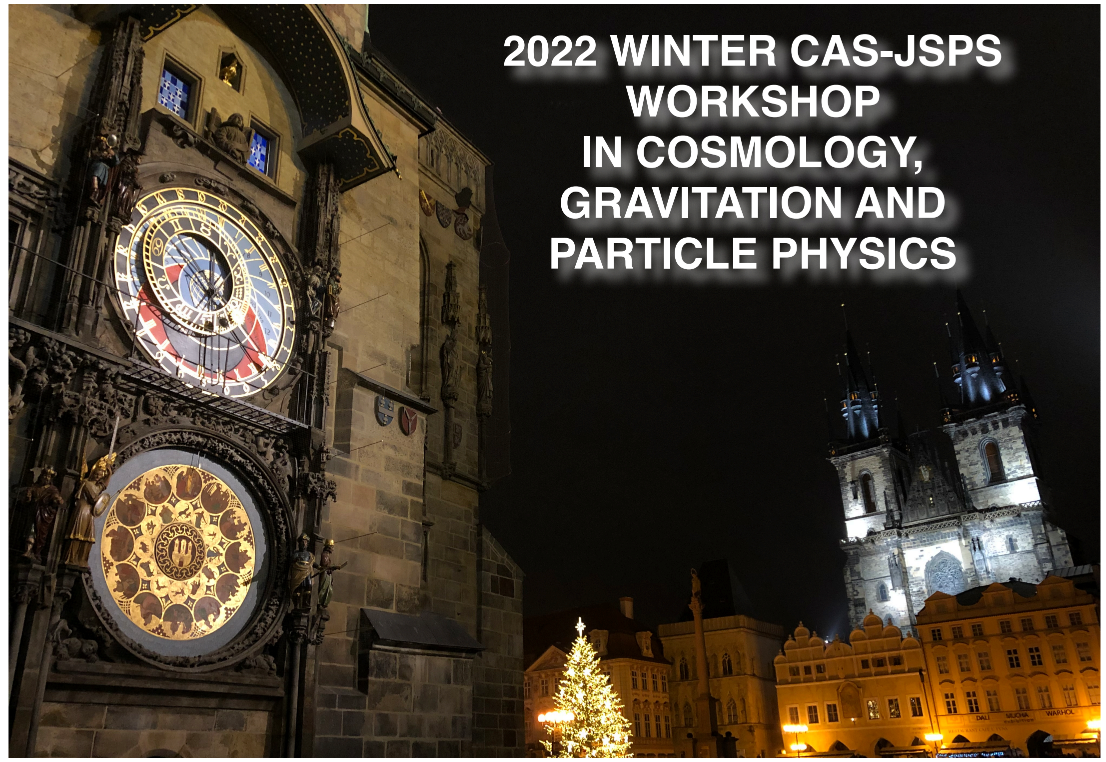

Living with Ghosts

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CEICO

EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research,
Development and Education

MŠMT
MINISTRY OF EDUCATION,
YOUTH AND SPORTS

Ghosts

are dynamical degrees of freedom
with negative mass
or kinetic energy unbounded from below

PHYSICAL REVIEW D, VOLUME 65, 103515

Not that kind of **Living with ghosts**

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(Received 27 July 2001; published 9 May 2002)

Perturbation theory for gravity in dimensions greater than two requires higher derivatives in the free action. Higher derivatives seem to lead to ghosts, ~~states with negative norm.~~ We consider a fourth order scalar field theory and show that the problem with ghosts arises because, in the canonical treatment, ϕ and $\Box\phi$ are regarded as two independent variables. Instead, we base quantum theory on a path integral, evaluated in Euclidean space and then Wick rotated to Lorentzian space. The path integral requires that quantum states be specified by the values of ϕ and $\phi_{,\tau}$. To calculate probabilities for observations, one has to trace out over $\phi_{,\tau}$ on the final surface. Hence one loses unitarity, but one can never produce a negative norm state or get a negative probability. It is shown that transition probabilities tend toward those of the second order theory, as the coefficient of the fourth order term in the action tends to zero. Hence unitarity is restored at the low energies that now occur in the universe.

PHYSICAL REVIEW LETTERS **128,** 041301 (2022)

Ghosts without Runaway Instabilities

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(Received 26 August 2021; accepted 24 December 2021; published 24 January 2022)

We present a simple class of mechanical models where a canonical degree of freedom interacts with another one with a negative kinetic term, i.e., with a ghost. We prove analytically that the classical motion of the system is completely stable for all initial conditions, notwithstanding that the conserved Hamiltonian is unbounded from below and above. This is fully supported by numerical computations. Systems with negative kinetic terms often appear in modern cosmology, quantum gravity, and high energy physics and are usually deemed as unstable. Our result demonstrates that for mechanical systems this common lore can be too naive and that living with ghosts can be stable.

e-Print: 2108.06294

DOI: 10.1103/PhysRevLett.128.041301

Less Stable Options

PHYSICAL REVIEW D **103,** 115025 (2021)

arXiv:2007.05541

Is negative kinetic energy metastable?

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Local minima of the potential can be metastable up to cosmologically long times thanks to energy conservation. We explore the possibility that theories with negative kinetic energy (ghosts) can be metastable up to cosmologically long times. In classical mechanics, ghosts undergo spontaneous lockdown rather than run away if weakly coupled and nonresonant. Physical examples of this phenomenon are shown. In quantum mechanics, this leads to metastability similar to vacuum decay. In classical field theory, lockdown is broken by resonances and ghosts behave statistically, drifting toward infinite entropy as no thermal equilibrium exists. We analytically and numerically compute the runaway rate finding that it is cosmologically slow in four-derivative gravity, where ghosts have gravitational interactions only. In quantum field theory, the ghost runaway rate is naively infinite in perturbation theory, analogously to what is found in early attempts to compute vacuum tunnelling; we do not know the true rate.

DOI: 10.1103/PhysRevD.103.115025

PHYSICAL REVIEW D **105,** 045018 (2022)

arXiv:2110.11175

Dynamical systems with benign ghosts

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(Received 18 November 2021; accepted 8 February 2022; published 23 February 2022)

We consider finite and infinite-dimensional *ghost-ridden* dynamical systems whose Hamiltonians involve nonpositive definite kinetic terms. We point out the existence of three classes of such systems where the ghosts are *benign*, i.e., systems whose evolution is unlimited in time: (i) systems obtained from the variation of bounded-motion systems; (ii) systems describing motions over certain Lorentzian manifolds and (iii) higher-derivative models related to certain modified Korteweg-de Vries equations.

DOI: 10.1103/PhysRevD.105.045018

Why are we interested in ghosts?

- Interesting cosmology: Phantom Dark Energy - super accelerated universe with $\dot{H} > 0$, Bouncing universe etc. Inflation with blue spectrum of gravity waves
- Higher derivative systems have ghosts due to the Ostrogradsky theorem (1848)
- jerk ($\dddot{x}$) equations are the minimal setting for solutions showing *chaotic behaviour* (electronic circuits (!) e.g. Chua's circuit)
- Renormalisation / quantisation of Gravity (Stelle, 1977)

$$S = \int M_{\mathrm{Pl}}^2 R + \alpha R^2 + \beta W_{\mu\nu\sigma\lambda} W^{\mu\nu\sigma\lambda}\,, \quad \text{Weyl tensor} \quad W = \partial\partial g$$

- Questions related to entropy and thermodynamics
- Is it possible to screen gravity?
- Is it possible to screen the Cosmological Constant or the energy of quantum vacuum?
- Can gravitons be massive? (Boulware–Deser ghost, 1972, dRGT etc.)

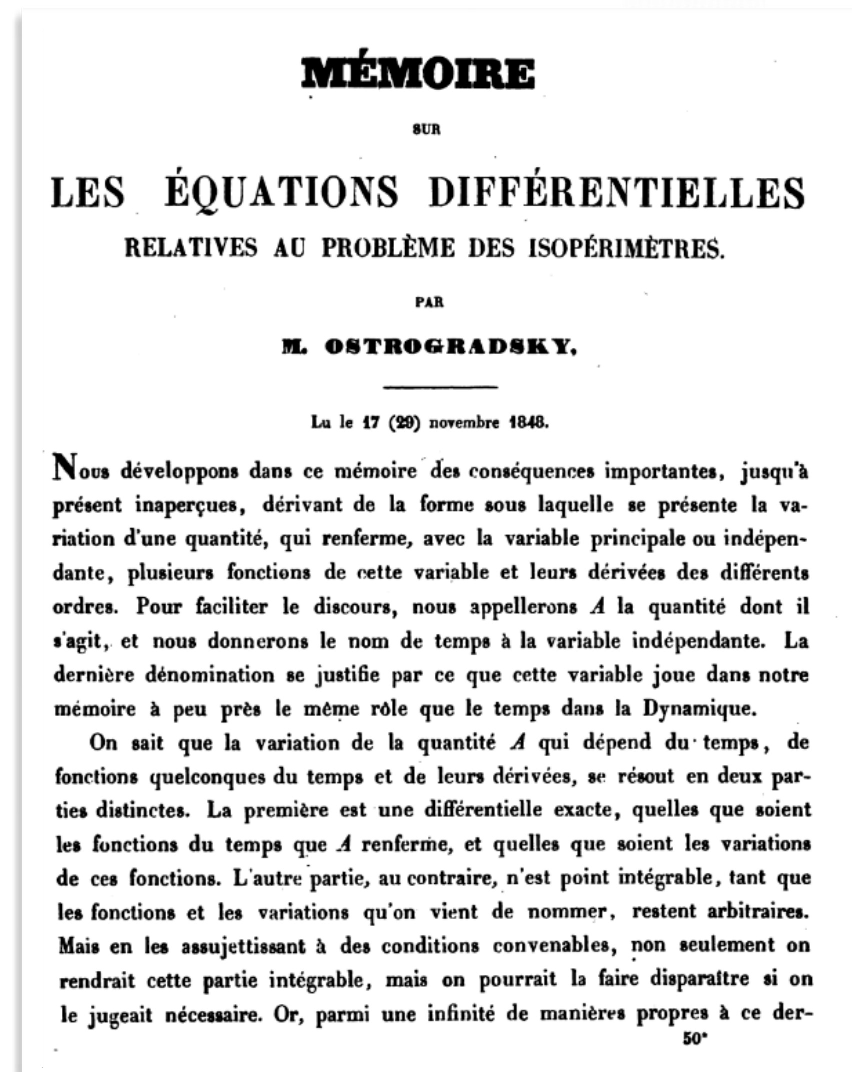
Ostrogradsky Theorem

modern version for poor people

$$\frac{1}{M^2p^2 - p^4} = \frac{1}{M^2}\left[\frac{1}{p^2} - \frac{1}{p^2 - M^2}\right]$$

propagator

Mikhail Vasilyevich Ostrogradsky



MÉMOIRE

SUR

LES ÉQUATIONS DIFFÉRENTIELLES

RELATIVES AU PROBLÈME DES ISOPÉRIMÈTRES.

PAR

M. OSTROGRADSKY.

Lu le 17 (29) novembre 1848.

Nous développons dans ce mémoire des conséquences importantes, jusqu'à présent inaperçues, dérivant de la forme sous laquelle se présente la variation d'une quantité, qui renferme, avec la variable principale ou indépendante, plusieurs fonctions de cette variable et leurs dérivées des différents ordres. Pour faciliter le discours, nous appellerons A la quantité dont il s'agit, et nous donnerons le nom de temps à la variable indépendante. La dernière dénomination se justifie par ce que cette variable joue dans notre mémoire à peu près le même rôle que le temps dans la Dynamique.

On sait que la variation de la quantité A qui dépend du temps, de fonctions quelconques du temps et de leurs dérivées, se résout en deux parties distinctes. La première est une différentielle exacte, quelles que soient les fonctions du temps que A renferme, et quelles que soient les variations de ces fonctions. L'autre partie, au contraire, n'est point intégrable, tant que les fonctions et les variations qu'on vient de nommer, restent arbitraires. Mais en les assujettissant à des conditions convenables, non seulement on rendrait cette partie intégrable, mais on pourrait la faire disparaître si on le jugeait nécessaire. Or, parmi une infinité de manières propres à ce der-

50*

For Lagrangian $L\left(q, \dot{q}, \ddot{q}\right)$ depending on acceleration $a = \ddot{q}$

canonical momentum for $Q_1 = q$

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$$

canonical momentum for $Q_2 = \dot{q}$

$$P_2 = \frac{\partial L}{\partial \ddot{q}}$$

$$H = P_1\dot{Q}_1 + P_2\dot{Q}_2 - L$$

$$H = P_1Q_2 + P_2 a\left(P_2, Q_1, Q_2\right) - L\left(Q_1, Q_2, a\left(P_2, Q_1, Q_2\right)\right)$$

Hamiltonian linear in P_1 - unbounded from above and from below!

Mechanical analogy

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

e.g. action for cosmological perturbations

$$S[\mathcal{R}] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, Z\left((\mathcal{R}')^2 - c_{\mathrm{S}}^2 (\partial_i \mathcal{R})^2\right)$$

$$H_{\mathbf{k}} = \frac{|P_{\mathbf{k}}|^2}{2Z} + \frac{Z c_S^2 k^2 |\mathcal{R}_{\mathbf{k}}|^2}{2}$$

$$Z \leftrightarrow m \qquad \omega^2 \leftrightarrow c_s^2 k^2$$

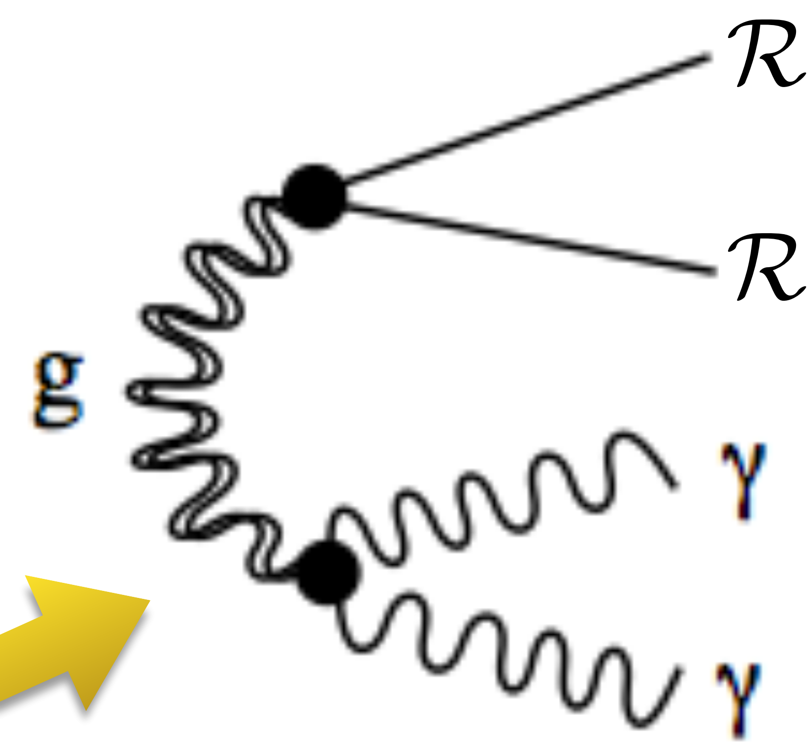
Ghosts and gradient instabilities

$$S\left[\mathcal{R}\right] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, Z\left(\left(\mathcal{R}'\right)^2 - c_\mathrm{S}^2\left(\partial_i\mathcal{R}\right)^2\right)$$

$$H_\mathbf{k} = \frac{\left|P_\mathbf{k}\right|^2}{2Z} + \frac{Z c_S^2 k^2 \left|\mathcal{R}_\mathbf{k}\right|^2}{2}$$

$$R_\mathbf{k} \sim \exp\left(\left|c_s\right| k\tau\right)$$

Gradient instability $c_s^2 < 0$



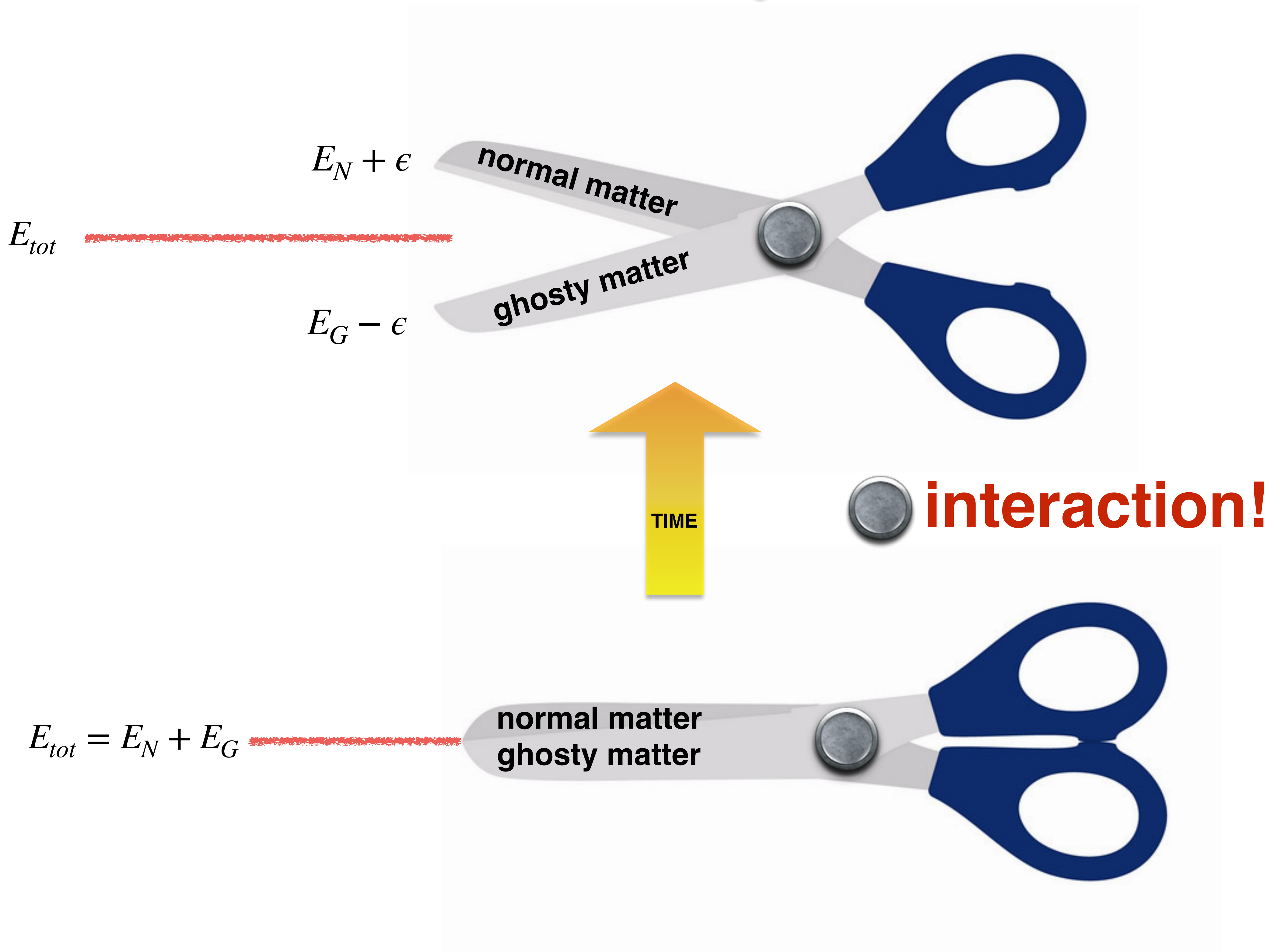
ghost $Z(t) < 0$

*ghosts - modes (oscillators) with the **negative mass***

$$\Gamma_{0\to 2\gamma 2\phi} \sim \frac{\Lambda^8}{M_\mathrm{Pl}^4}$$

Cline, Jeon, Moore, (2003)

Instability



How Unstable?

$$H = \frac{P^2}{2} + \frac{Q^2}{2} - \left(\frac{p^2}{2} + \frac{\omega^2 q^2}{2} \right) + \frac{\lambda}{2} q^2 Q^2$$

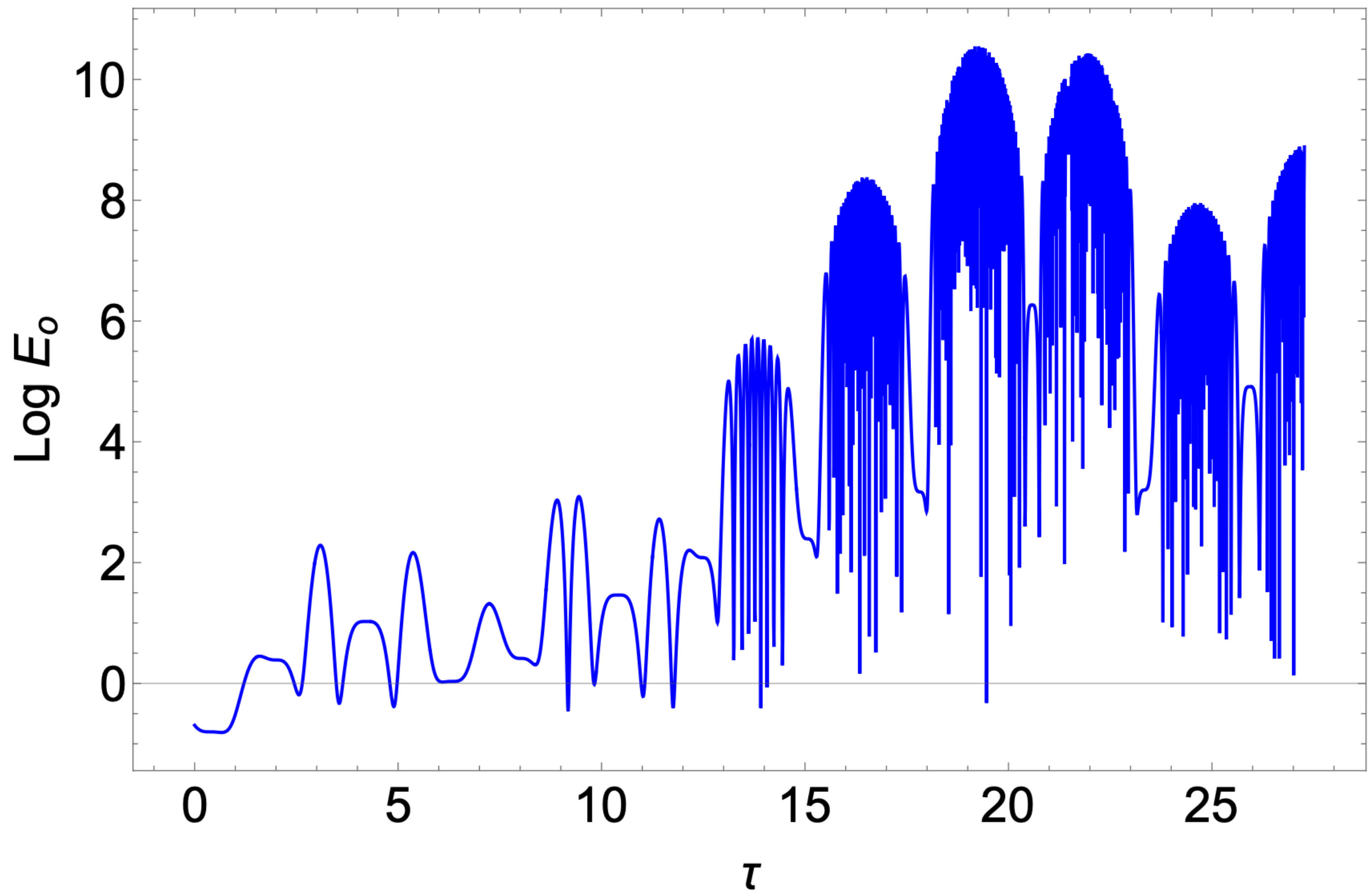


Figure 2: The growth of the logarithm of the energy of the observer is depicted for $\lambda = 4$, $\omega = 2.3$ and vacuum initial data (8) and (9).

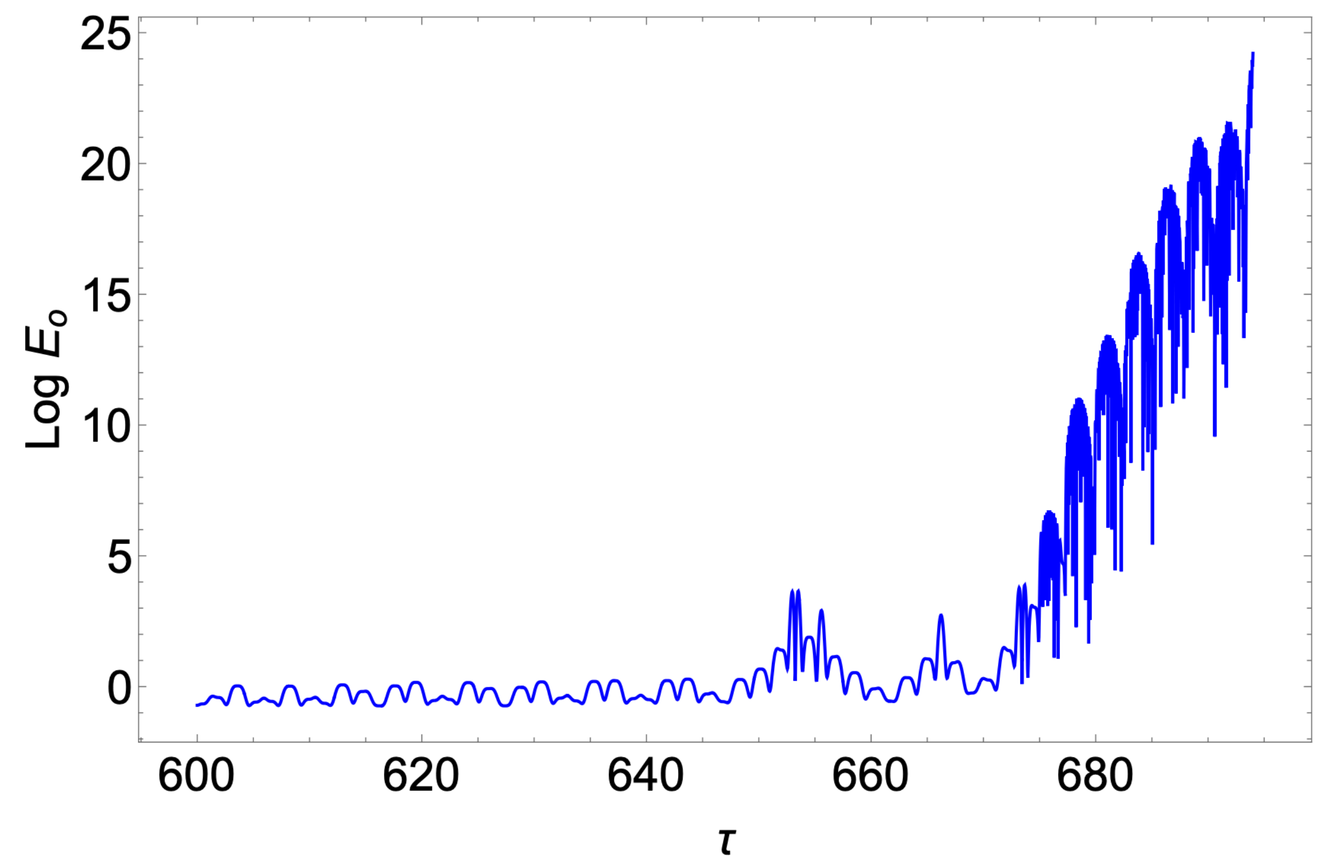


Figure 3: The growth of the logarithm of the energy of the observer is depicted for $\lambda = 2.35$, $\omega = 2.3$ and vacuum initial data (8) and (9). Here we see that the instability arises only much later after around a 100 of the periods of oscillation for the observer.

Our Model

Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^2) - \frac{1}{2}(p_y^2 + y^2) + V_I(x, y)$$

Normal Oscillator Ghosty Oscillator Interaction Potential

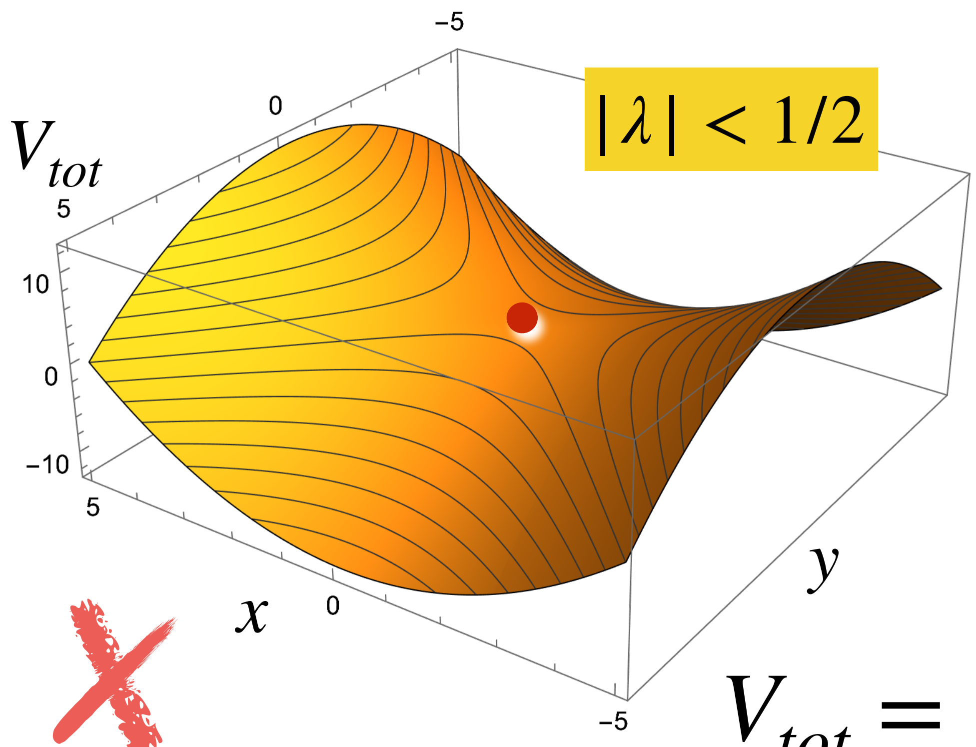
$$V_I(x, y) = \lambda \left[1 + 2\left(y^2 + x^2\right) + \left(y^2 - x^2\right)^2\right]^{-1/2}$$

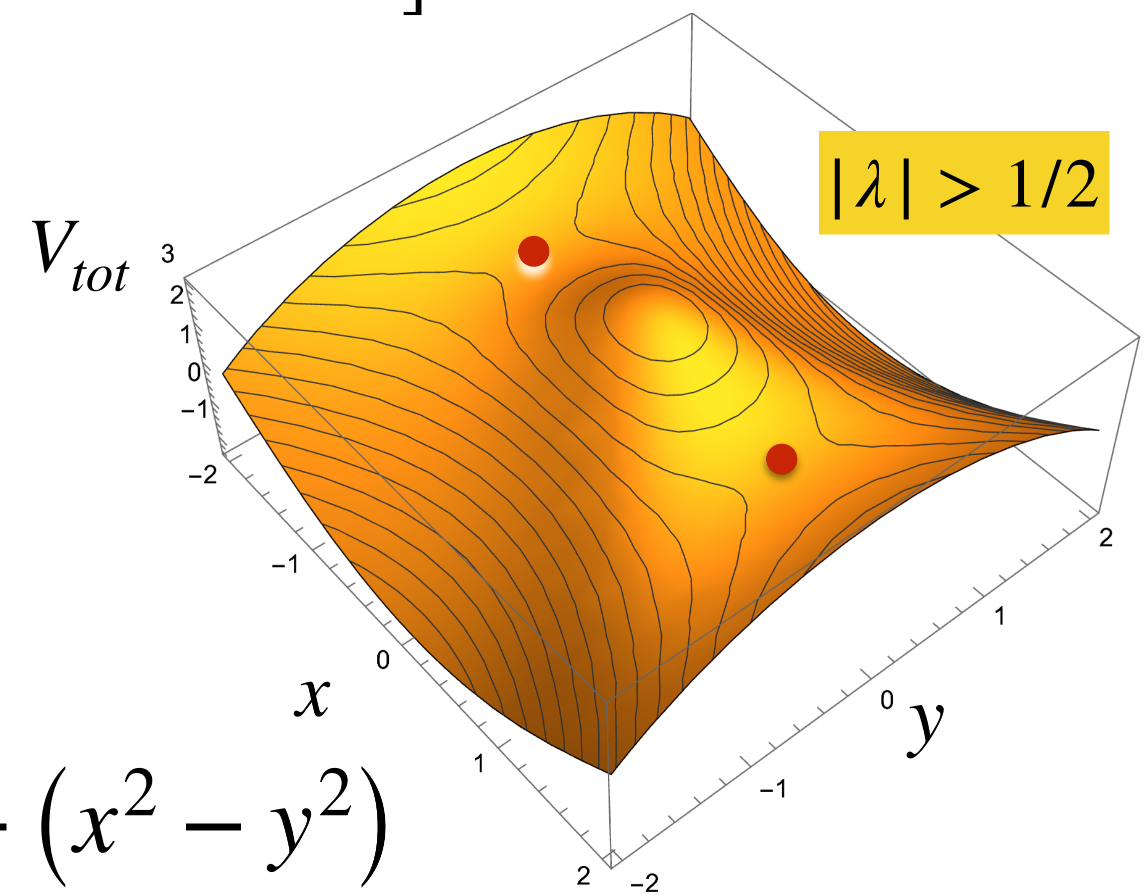
Coupling Constant

Interaction is bounded $0 < V_I(x, y)\, \lambda^{-1} \leq 1$

Potential

$$V_I(x,y) = \lambda\left[\left(x^2 - y^2 - 1\right)^2 + 4x^2\right]^{-1/2}$$





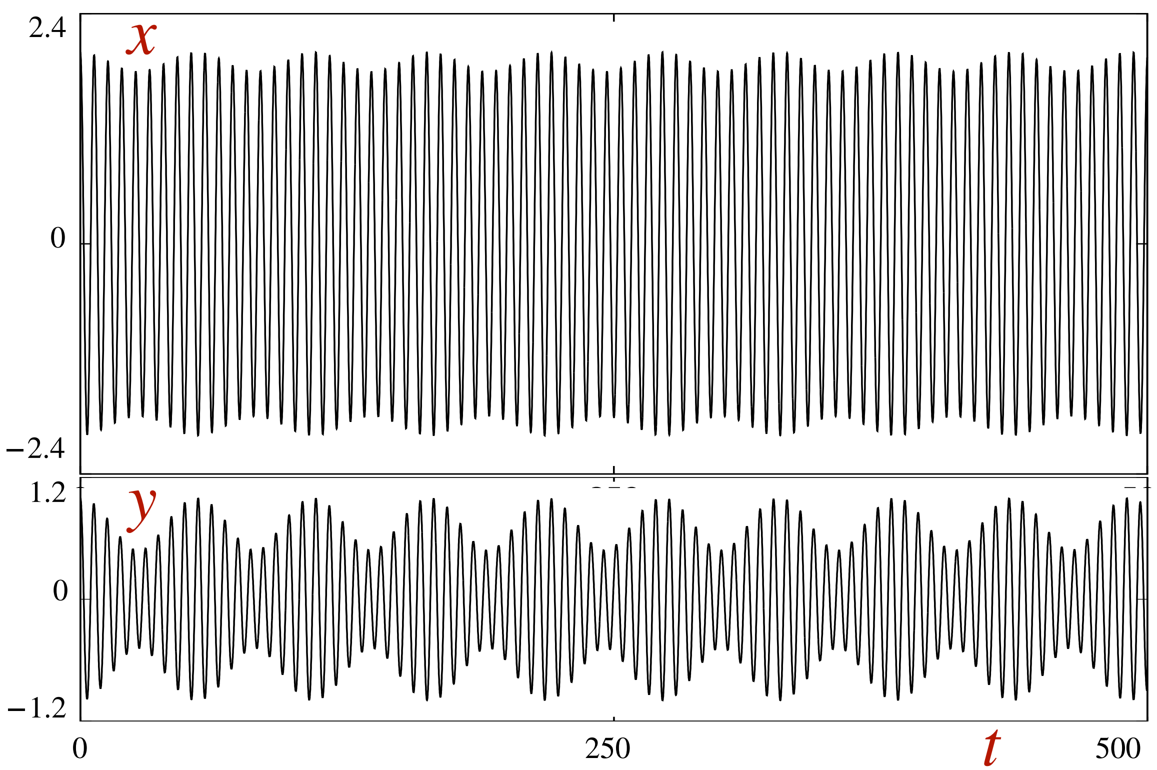
$$V_{tot} = V_I + \frac{1}{2}\left(x^2 - y^2\right)$$

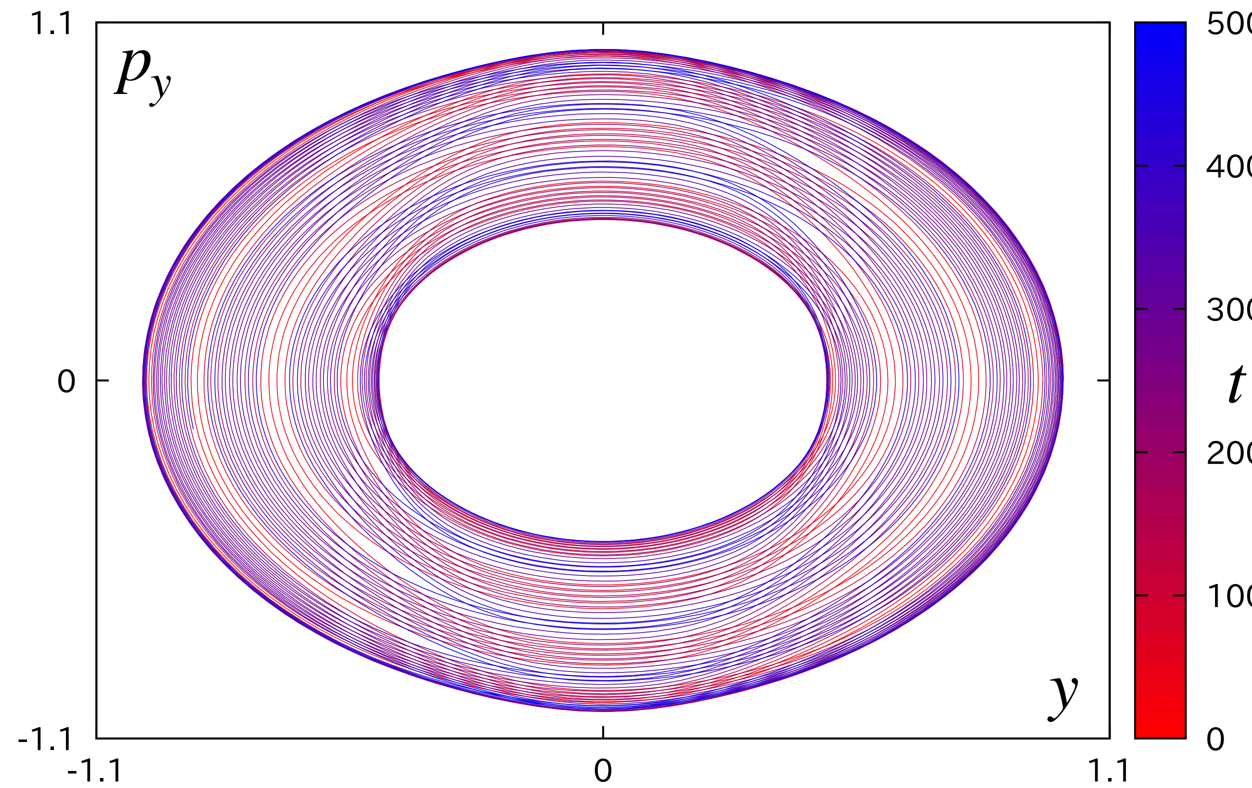
$$V_{tot} = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda\left(x^4 + 4y^2x^2 + y^4\right) + \ldots$$

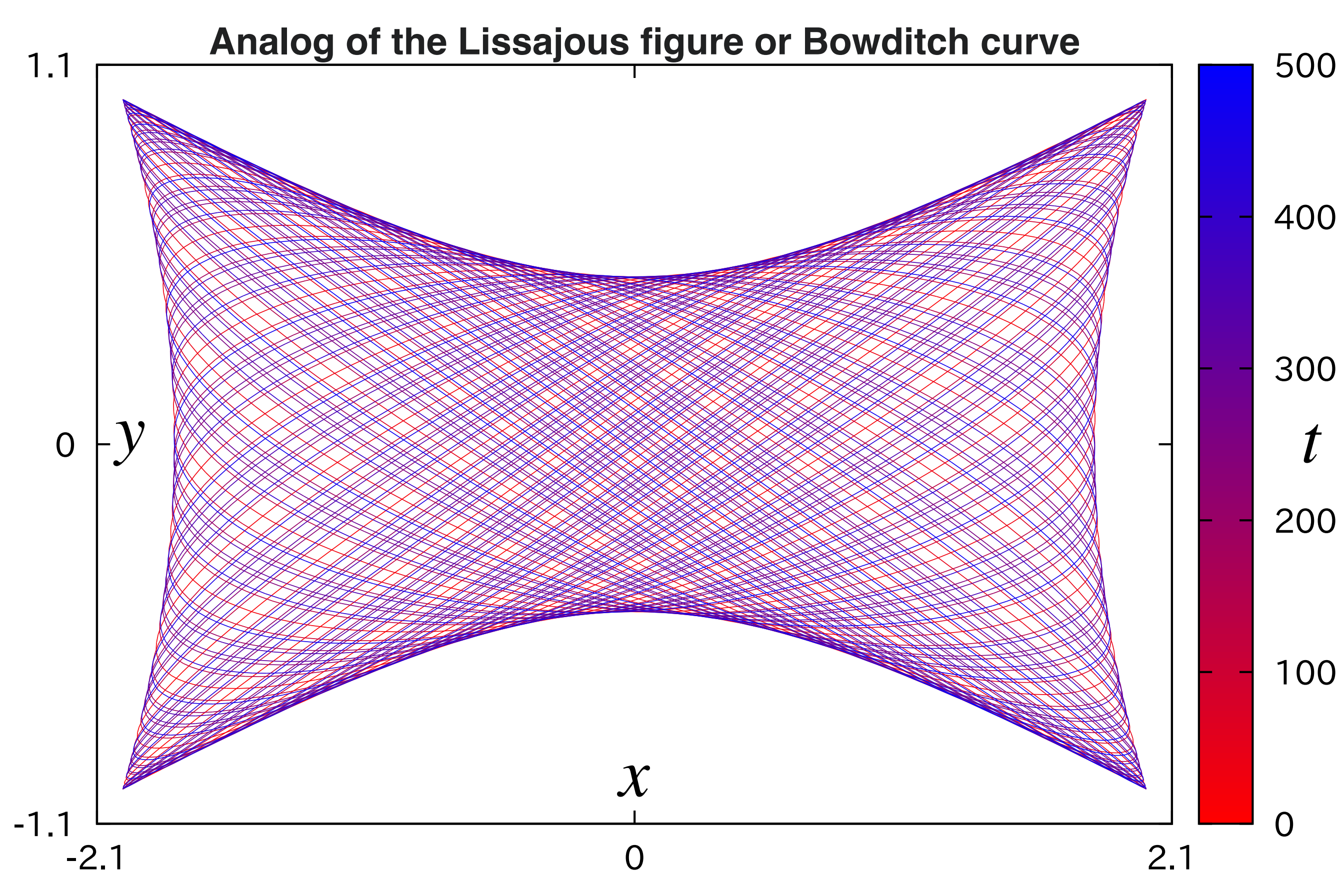
$\omega_x^2 = 1 - 2\lambda$, and $\omega_y^2 = 1 + 2\lambda$

Stable motion not at a minimum, but at a saddle point of the potential!

Numerical Solutions







Numerical solutions look pretty stable
for hundreds of oscillations,
and some initial data.

Maybe it is just a stable system?

Can one analytically prove stability
and the absence of the runaway solutions?

First Integral and the Power of Imagination

$$C = K^2 + \left(p_x^2 + x^2\right) - \left(x^2 - y^2 - 1\right) V_I\left(x, y\right)$$

generator for hyperbolic rotations $K = p_y x + p_x y$

$$\frac{dC}{dt} = 0$$

One can obtain our system via complex canonical

transformation $y = i\tilde{y}$, and $p_y = -i\tilde{p}_y$

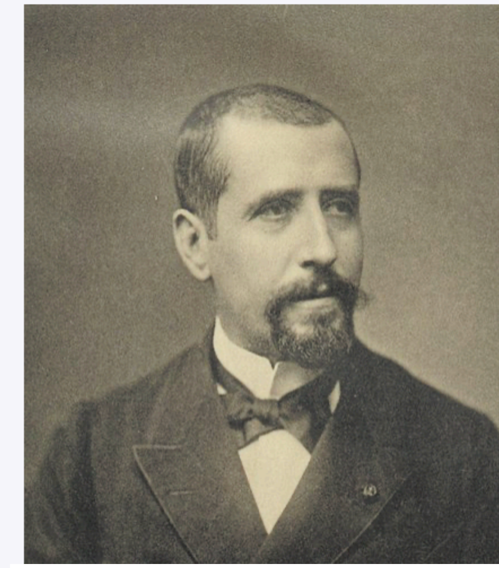
(so that $[y, p_y] = [\tilde{y}, \tilde{p}_y] = 1$ etc.)

from a ghost-free integrable system introduced by

Darboux in 1901

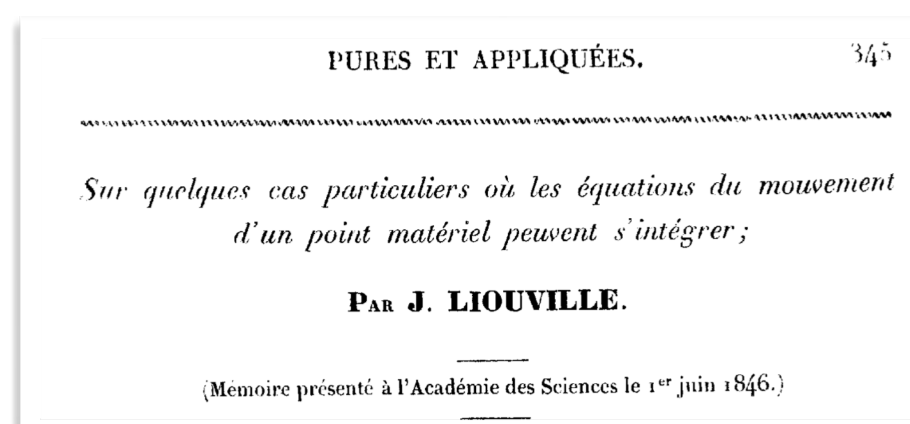
Joseph Liouville 1846

Is there any symmetry behind this conserved quantity C?



Jean-Gaston Darboux
FAS MIF FRS FRSE

Joseph Liouville
FRS FRSE FAS



PURES ET APPLIQUÉES. 345

Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer;

PAR J. LIOUVILLE.

(Mémoire présenté à l'Académie des Sciences le 1er juin 1846.)

Another First Integral: $\mathscr{E}$

$$\mathscr{E} = C - H = \Sigma + \left(y^2 - x^2\right) V_I\left(x, y\right)$$

where

$$\Sigma = \left(p_y x + p_x y\right)^2 + \frac{1}{2}\left(p_x^2 + x^2\right) + \frac{1}{2}(p_y^2 + y^2)$$

the interaction part is always bounded

$$-|\lambda| \leq \left(y^2 - x^2\right) V_I\left(x, y\right) \leq |\lambda|$$

as

$$V_I\left(x, y\right) = \lambda\left[1 + 2\left(y^2 + x^2\right) + \left(y^2 - x^2\right)^2\right]^{-1/2}$$

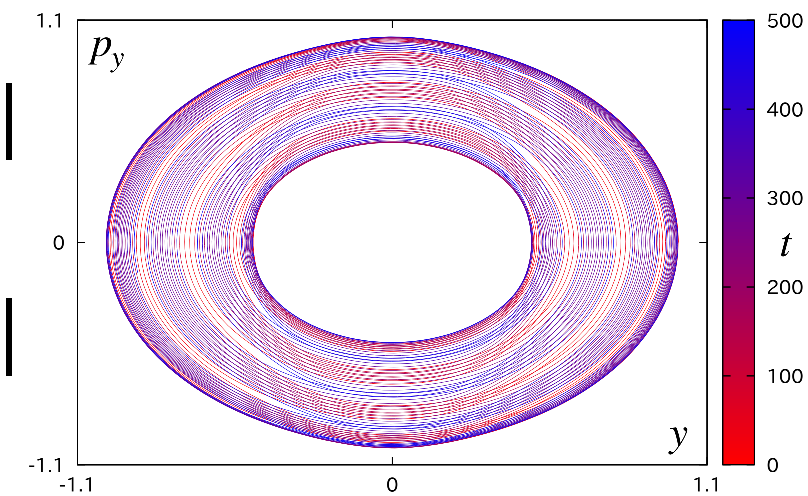
⇓

for all times $\Sigma - |\lambda| \leq \mathscr{E} \leq \Sigma + |\lambda|$

Finiteness of motion

at initial point of time t_a $\Sigma_a - |\lambda| \leq \mathscr{E} \leq \Sigma_a + |\lambda|$

at any later point in time t_b $\Sigma_b - |\lambda| \leq \mathscr{E} \leq \Sigma_b + |\lambda|$



Σ is positive definite and
is confined in a stripe

$$\Sigma_a - 2|\lambda| \leq \Sigma_b \leq \Sigma_a + 2|\lambda|$$

Thus the trajectory is confined in a stripe, as for $\xi = (x, y, p_x, p_y)$ we have $|\xi|^2 \leq 2\Sigma$

System always evolves in a finite region of phase space

Lyapunov Stability

$$\mathscr{E} = C - H = \Sigma + \left(y^2 - x^2\right) V_I\left(x, y\right)$$

where

$$\Sigma = K^2 + \frac{1}{2}\left(p_x^2 + x^2\right) + \frac{1}{2}(p_y^2 + y^2)$$

- at the origin $\mathscr{E}(0) = 0$

- for $\lambda\left(y^2 - x^2\right) > 0$ this first integral is positive, $\mathscr{E} > 0$

- for $\lambda\left(y^2 - x^2\right) < 0$ this first integral

$$\mathscr{E} > \Sigma + \lambda\left(y^2 - x^2\right) = K^2 + \frac{1}{2}\left(p_x^2 + p_y^2 + \omega_x^2 x^2 + \omega_y^2 y^2\right) > 0 \text{ for } |\lambda| < 1/2$$

$\mathscr{E}$ is a Lyapunov function
so that the system is stable at the origin for $|\lambda| < 1/2$

Aleksandr Mikhailovich Lyapunov

The General Problem of the Stability of Motion, Doctoral dissertation, Kharkov U. 1892

Kolmogorov–Arnold–Moser (KAM) theorem

Small structural changes
do not jeopardise
the stability and finiteness
of motion

Realisation through Higher Derivatives

$$L(q, \ddot{q}) = \left(\ddot{q} + q\right)\left(2p_2 + (2p_2)^{-1}\right)$$

where $p_2 \equiv p_2\left(q, \ddot{q}\right)$ is the solution of

$$(\ddot{q} + q)\sqrt{2q^2 + 1} = -\, 2\lambda p_2(2p_2^2 + 1)^{-3/2}$$

In this way $p_2 = \partial L / \partial \ddot{q}$

Why have not we seen
such systems so far in nature?


拝観謝絶
西源院
Thanks a lot for attention!